COMMUTATIVITY OF OPERATOR TUPLES

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ABSTRACT. For an n-tuple $\mathbb{A} = (A_1, \cdots, A_n)$ of compact operators we define the joint point spectrum of $\mathbb{A}$ to be the set

$$\sigma_p(\mathbb{A}) = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : \ker(I + z_1 A_1 + \cdots + z_n A_n) \neq (0)\}.$$

We prove in several situations that the operators in $\mathbb{A}$ pairwise commute if and only if $\sigma_p(\mathbb{A})$ consists of countably many, locally finite, hyperplanes in $\mathbb{C}^n$. In particular, we show that if $\mathbb{A}$ is an n-tuple of $N \times N$ normal matrices, then these matrices pairwise commute if and only if the polynomial

$$p_{\mathbb{A}}(z_1, \cdots, z_n) = \det(I + z_1 A_1 + \cdots + z_n A_n)$$

is completely reducible, namely,

$$p_{\mathbb{A}}(z_1, \cdots, z_n) = \prod_{k=1}^{N} (1 + a_{k1} z_1 + \cdots + a_{kn} z_n)$$

can be factored into the product of linear polynomials.